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SU(2) Charges as Angular Momentum in *N* = 1 Self-Dual Supergravity

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The $N = 1$ self-dual supergravity has $SL(2, \mathbb{C})$ symmetry. This symmetry results in $SU(2)$ charges as the angular momentum. As in nonsupersymmetr ic self-dual gravity, the currents are also of their potentials and are therefore identically conserved. The charges are generally invariant and gauge covariant under local $SU(2)$ transforms and approach being rigid at spatial infinity. The Poisson brackets constitute the $su(2)$ algebra and hence can be interpreted as the generally covariant conservative angular momentum.

The study of self-dual gravities has drawn much attention in the past decade since the discovery of Ashtekar's new variables, in terms of which the constraints can be greatly simplified.[1,2] The new phase variables consist of densitized $SU(2)$ soldering forms $\tilde{e}^i\ A^B$ from which a metric density is obtained according to the definition $q_{ij} = -\mathrm{Tr}\tilde{e}_i\tilde{e}_j$, and a complexified connection A_{iA}^B which carries the momentum dependence in its imaginary part. Ashtekar's original self-dual canonical gravity also permits a Lagrangian formulation.[3,4] The supersymmetric extension of this Lagrangian formulation, which is equivalent to simple real supergravity, was proposed by Jacobson,[5] and the corresponding Ashtekar complex canonical transform was given by Gorobey and Lukyanenko.[6] The Lagrangian density is[5]

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$$\mathcal{L}_j = \frac{1}{\sqrt{2}} (e^{AA'} \wedge e_{BA'} \wedge F_A{}^B + ie^{AA'} \wedge \bar{\psi}_{A'} \wedge \mathcal{D}\psi_A \tag{1}$$

The dynamical variables are the real tetrad $e^{AA'}$ ("real" means $\bar{e}^{A'A} = e^{AA'}$), the traceless left-handed $SL(2, \mathbb{C})$ connection $A_{\mu MN}$, and the complex anticommuting spin-3/2 gravitino field $\psi_{\mu A}$. The $SL(2, \mathbb{C})$ covariant exterior derivative is defined by

$$\mathcal{D}\psi_M := d\psi_M + A_M{}^N \wedge \psi_N \tag{2}$$

and the curvature 2-form is

$$F_M{}^N := dA_M{}^N + A_M{}^P \wedge A_P{}^N \tag{3}$$

The indices are lowered and raised with the antisymmetric $SL(2, \mathbb{C})$ spinor ϵ^{AB} and its inverse ϵ_{AB} according to the convention $\lambda_B = \lambda^A \epsilon_{AB}$, $\lambda^A = \epsilon^{AB}\lambda_B$, and the implied summations are always in northwesterly fashion: from the left-upper to the right-lower. The Lagrangian equation (1) is a holomorphic function of the connection, and the equation for $A^B_{\mu A}$ is equivalent to

$$\mathcal{D}e^{AA'} = \frac{i}{2} \psi^A \wedge \bar{\psi}^{A'} \tag{4}$$

provided $e^{AA'}$ is real. The Lagrangian $\frac{1}{2}(\mathcal{L}_J + \bar{\mathcal{L}}_J)$ for real supergravity is a nonholomorphic function, but leads to no surfeit of field equations. Under the left-handed local supersymmetric transform generated by the anticommuting parameters ϵ_A

$$\delta\psi_A = 2\mathcal{D}\epsilon_A, \qquad \delta\bar{\psi}_{A'} = 0, \qquad \delta e_{AA'} = -i\bar{\psi}_{A'}\epsilon_A \tag{5}$$

the Lagrangian $\mathcal{L}_J$ is invariant *without* using any of the Euler–Lagrangian equations, while under the right-handed transform

$$\delta\psi_A = 0, \qquad \delta\bar{\psi}_{A'} = 2\mathcal{D}\bar{\epsilon}_{A'}, \qquad \delta e_{AA'} = -i\psi_A\bar{\epsilon}_{A'} \tag{6}$$

$\mathcal{L}_J$ is invariant *modulo* the field equations.

The (3 + 1) decomposition is effected as

$$\mathcal{L}_j = \tilde{e}^{kAB}\dot{A}_{kAB} + \tilde{\pi}^{KA}\dot{\psi}_{kA} - \mathcal{H} \tag{7}$$

$$\mathcal{H} := e_{0AA'}\,\mathcal{H}^{AA'} + \psi_{0A}\mathcal{S}^A + \hat{\mathcal{S}}^{A'}\bar{\psi}_{0A'} + A_{0AB}\mathcal{J}^{AB} + (\text{total divergence}) \tag{8}$$

The canonical momenta are

$$\tilde{e}^{kAB} := -\frac{1}{\sqrt{2}} \epsilon^{ijk} e_i{}^{AA'} e^B_{jA'} \tag{9}$$

$$\tilde{\pi}^{kA} := \frac{i}{\sqrt{2}} \epsilon^{ijk} e_i{}^{AA'} \bar{\psi}_{jA'} \tag{10}$$

and the constraints are

$$\mathcal{H}^{AA'} := \frac{1}{\sqrt{2}} \epsilon^{ijk}(e_i^{BA'} F_{jkB}^{A} - i\bar{\psi}_i{}^{A'} \mathcal{D}_j \psi_K{}^{A}) \tag{11}$$

$$\mathcal{S}^{A} := \mathcal{D}_k \tilde{\pi}^{kA} \tag{12}$$

$$\hat{\mathcal{S}}^{A'} := \frac{i}{\sqrt{2}} \epsilon^{ijk} e_i{}^{AA'} \mathcal{D}_j \psi_{kA} \tag{13}$$

$$\mathcal{J}^{AB} := \mathcal{D}_k \tilde{e}^{kAB} - \tilde{\pi}^{k(A} \psi_k^{B)} \tag{14}$$

The 0-components $e_{0AA'}$, ψ_{0A}, $\bar{\psi}_{0A'}$, and A_{0AB} are just the Lagrange multipliers and the dynamical conjugate pairs are $(\tilde{e}^{kAB}, A_{jAB})$, $(\tilde{\pi}^{kA}, \psi_{kA})$. The constraints $\mathcal{H}^{AA'} = 0$ and $\hat{\mathcal{S}}^{A'} = 0$ generate

$$\ddot{\mathcal{H}}^{AB} := (\tilde{e}^j \tilde{e}^k F_{jk})^{AB} + 2\tilde{\pi}^j \tilde{e}^k \mathcal{D}_{[j} \psi_{k]} \epsilon^{AB} + 2(\tilde{\pi}^j \mathcal{D}_{[j} \psi_{k]}) \tilde{e}^{kAB} = 0 \tag{15}$$

$$\mathcal{S}^{\dagger A} := \frac{1}{\sqrt{2}} \epsilon^{ijk} \tilde{e}_i{}^{AB} \mathcal{D}_j \psi_{kB} = 0 \tag{16}$$

The equations of motion will be properly expressed in Hamiltonian form $\dot{f} = \{H, f\}$ if we assign the Poisson brackets

$$\{\tilde{e}^{kAB}(x), A_{jAB}(y)\} = \delta_j^k \delta_{(M}^A \delta_{N)}{}^B \delta^3(x, y) \tag{17}$$

$$\{\tilde{\pi}^{kA}(x), \psi_{jA}(y)\} = -\delta_j^k \delta_M{}^A \delta^3(x, y) \tag{18}$$

all other brackets among these quantities being zero. This is the outline of the theory.

In previous work we obtained the $SU(2)$ charges and the energy-momentum in the Ashtekar formulation of Einstein gravity[7,8] and they are closely related to the angular momentum[9–11] and the energy-momentum[12] in the vierbein formalism of Einstein gravity. The fact that the algebra formed by their Poisson brackets does constitute the 3-Poincaré algebra on the Cauchy surface supports from another aspect that their definitions are reasonable. Similarly, the study of $SU(2)$ charges in the self-dual supergravity considered is also an interesting subject. In the following, we will employ the $SL(2, \mathbb{C})$ invariance to obtain conservative charges as we did previously.[8] Under any $SL(2, \mathbb{C})$ transform

$$e_{\mu AA'} \to L_A^B \bar{R}_{A'}{}^{B'} e_{\mu BB'}, \qquad \psi_A \to L_A{}^B \psi_B, \qquad \bar{\psi}_{A'} \to \bar{R}_{A'}{}^{B'} \bar{\psi}_{B'}$$

$$A_{\mu MN} \to L_M{}^A A_{\mu A}{}^B (L^{-1})_{BN} + L_M{}^A \partial_\mu (L^{-1})_{AN} \tag{19}$$

$\mathcal{L}_J$ is invariant. L and $\bar{R}$ are not necessarily related by complex conjugation. Note that $L_{AB} = -(L^{-1})_{BA}$; the transform of A may also be written as

$$A_{\mu MN} \rightarrow L_M{}^A L_N{}^B A_{\mu AB} - L_M{}^A \partial_\mu L_{NA} \tag{20}$$

For infinitesimal transform, $L_A{}^B = \delta_A{}^B + \xi_A{}^B$, where $\xi_{AB} = -\xi_{BA}$ are infinitesimal parameters. Thus we have

$$\delta_\xi A = [\xi, A] - d\xi, \qquad \delta\psi = \xi\psi \tag{21}$$

When calculating the variation of the Lagrangian, one must take into consideration the anticommuting feature of the gravitino field. We write the variation in such a way that

$$\delta\mathcal{L}_J = \delta\phi^A\left(\frac{\partial}{\partial\phi^A} - \partial_\mu \frac{\partial}{\partial\partial_\mu\phi^A}\right)\mathcal{L}_J + \partial_\mu\left(\delta\phi^A \frac{\partial}{\partial\partial_\mu\phi^A}\mathcal{L}_J\right) \tag{22}$$

where ϕ^A denotes any field involved in the first-order Lagrangian. Now both $\partial/\partial\phi^A$ and $\partial/\partial\partial_\mu\phi^A$ are (anti)commuting if ϕ^A is (anti)commuting, and so there is no ordering problem.

The invariance of $\mathcal{L}_J$ under the infinitesimal $SL(2, \mathbb{C})$ transform is equivalent to the following *modulo* the field equations:

$$\partial_\rho\left(\delta A_{\sigma A}{}^B \frac{\partial\mathcal{L}_J}{\partial\partial_\rho A_{\sigma A}{}^B} + \delta\psi_{\sigma A}\frac{\partial\mathcal{L}_J}{\partial\partial_\rho\psi_{\sigma A}}\right) = 0 \tag{23}$$

For constant ξ, we have

$$\partial_\rho\left(\frac{1}{\sqrt{2}}\epsilon^{\mu\nu\rho\sigma}e_\mu{}^{AA'}e_{\nu BA'}[\xi, A_\sigma]_A{}^B + \frac{i}{\sqrt{2}}\epsilon^{\mu\nu\rho\sigma}e_\mu{}^{AA'}\bar{\psi}_{\nu A'}(\xi\psi_\sigma)_A\right) = 0 \tag{24}$$

We therefore have the conservation of $SU(2)$ charges

$$\partial_\mu \tilde{j}^\mu_{AB} = 0 \tag{25}$$

where

$$\tilde{j}^\rho_{AB} = \frac{1}{\sqrt{2}}\epsilon^{\mu\nu\rho\sigma}\Big(e_{\mu A}{}^{A'}e_{\nu MA'}A_{\sigma B}{}^M - e_\mu{}^{MA'}e_{\nu BA'}A_{\sigma MA} + \frac{i}{2}e_{\mu A}{}^{A'}\bar{\psi}_{\nu A'}\psi_{\sigma B} + \frac{i}{2}e_{\mu B}{}^{A'}\bar{\psi}_{\nu A'}\psi_{\sigma A}\Big) \tag{26}$$

Thus

$$J_{AB} = \int_\Sigma \tilde{j}^0_{AB}\, d^3x \tag{27}$$

where

$$\tilde{j}^0_{AB} = \frac{1}{\sqrt{2}} \epsilon^{ijk} \Big(e_{iA}{}^{A'} e_{jMA'} A_{kB}{}^{M} - e_i^{MA'} e_{jBA'} A_{kMA} + \frac{i}{2} e_{iA}{}^{A'} \bar{\psi}_{jA'} \psi_{kB} + \frac{i}{2} e_{iB}{}^{A'} \bar{\psi}_{jA'} \psi_{kA} \Big) \tag{28}$$

Using Eqs. (9) and (10), we can write $\tilde{j}^0_{AB}$ as

$$\tilde{j}^0_{AB} = [\tilde{e}^k, A_k]_{AB} + \tilde{\pi}_{k(A} \psi^k_{B)} \tag{29}$$

The constraint $\mathcal{T}_{AB} = 0$ guarantees that

$$J_{AB} \approx \int_\Sigma \partial_k \tilde{e}^k_{AB} = \int_{\partial\Sigma} \tilde{e}^k_{AB}\, ds_i \tag{30}$$

where $ds_i = \frac{1}{2}\epsilon_{ijk}\, dx^j \wedge dx^k$. It can also be obtained in the following way. Using the field equation $e^{A'(A} \wedge (\mathcal{D} e^{B)}_{A'} - \frac{i}{2}\psi^{B)} \wedge \bar{\psi}_{A'}) = 0$, we have

$$\epsilon^{\rho\mu\nu\sigma} \Big[e^{A'}_{\mu A} \Big(\partial_\sigma e_{\nu BA'} + A^M_{\sigma B} e_{\nu MA'} + \frac{i}{2} \bar{\psi}_{\nu A'} \psi_{\sigma B} \Big) + e^{A'}_{\mu B} \Big(\partial_\sigma e_{\nu AA'} + A^M_{\sigma A} e_{\nu MA'} + \frac{i}{2} \bar{\psi}_{\nu A'} \psi_{\sigma A} \Big) \Big] = 0 \tag{31}$$

so

$$\tilde{j}^\rho_{AB} = -\frac{1}{\sqrt{2}} \epsilon^{\rho\mu\nu\sigma} \partial_\sigma (e^{A'}_{\mu A} e_{\nu BA'}) \tag{32}$$

Using

$$e_{[\mu A}{}^{A'} e_{\nu]BA'} = e_{[\mu AC} e_{\nu]B}{}^{C} - i\sqrt{2} n_{[\mu} e_{\nu]AB} \tag{33}$$

we have

$$\begin{aligned} \tilde{j}^0_{AB} &= -\frac{1}{\sqrt{2}} \epsilon^{ijk} \partial_k (e^{A'}_{[iA} e_{j]BA'}) \\ &= -\frac{1}{\sqrt{2}} \epsilon^{ijk} \partial_k (e_{[iAC} e_{j]B}{}^{C} - i\sqrt{2} n_{[i} e_{j]AB}) \\ &= \frac{1}{\sqrt{2}} \epsilon^{ijk} \partial_k (e_i e_j)_{AB} = \partial_k \tilde{e}^k_{AB} \end{aligned} \tag{34}$$

which is exactly the same as Eq. (30) We can thus have the Poisson brackets

$$\{J_{AB}, J_{MN}\} = \left\{\int_{\partial\Sigma} \tilde{e}^k_{AB}\, ds_k, \int_\Sigma (\tilde{e}^i_M{}^P A_{iPN} + \tilde{e}^i_N{}^P A_{iPN})\, d^3x\right\}$$
$$= \frac{1}{2}(J_{MA}\epsilon_{NB} + J_{MB}\epsilon_{NA} + J_{NA}\epsilon_{MB} + J_{MA}\epsilon_{NB}) \qquad (35)$$

Now the flat dreibein on Σ is needed in order to find the angular momentum J_i. To clarify the notions, we use the following conventions: $\mu, \nu, \ldots$ denote the 4-dimensional curved indices and i, j, k, denote the 3-dimensional curved indices on Σ; $a, b, c, \ldots$ denote the flat 4-dimensional indices and $l, m, n, \ldots$ denote the flat 3-dimensional indices on Σ. The rigid flat vierbein is denoted $E^a_{AA'}$ and the rigid flat dreibein is denoted E^m_{AB}. Then define

$$J_m := \frac{1}{\sqrt{2}} E^{AB}_m J_{AB} \qquad (36)$$

and using the relation $\epsilon^{mnl} E_m E_n = \sqrt{2} E^l$, we have

$$\{J_m, J_n\} = \epsilon_{mnl} J^l \qquad (37)$$

Therefore the $su(2)$ algebra is restored. As in the nonsupersymmetric case,[8] we can also obtain only the $SU(2)$ charges instead of the whole $SL(2, \mathbb{C})$ charges. Yet, the angular momentum J_{ab} obtained in refs. 9 and 10 is completely contained in J_{MN} since we have from Eq. (32) that

$$\tilde{j}^\rho_{AB} = -\frac{1}{2} \tilde{j}^\rho_{ab} E^a{}_A{}^{A'} E^b_{BA'} \qquad (38)$$

where $\tilde{j}^\rho_{ab}$ is the angular momentum current obtained in refs. 9 and 10,

$$\tilde{j}^\rho_{ab} = \sqrt{2}\epsilon^{\rho\sigma\mu\nu}\partial_\sigma(e_{\mu a} e_{\nu b}) \qquad (39)$$

and the angular momentum is

$$J_{ab} = \int_\Sigma \tilde{j}^0_{ab}\, d^3x \qquad (40)$$

Hence

$$J_{MN} = -\tfrac{1}{2} J^{ab} E_{[aM}{}^{A'} E_{b]NA'} = -\tfrac{1}{2}(J^{ij} E_{[iAC} E_{j]}{}^{BC} - i\sqrt{2} J^{0i} n_0 E_{iA}{}^B) \qquad (41)$$
$$= \frac{1}{\sqrt{2}}(L_i - iK_i) E^i_{MN}$$

where $L_i = \frac{1}{2}\epsilon_{ijk} J^{jk}$ are the spatial rotations and $K_i = J_{0i} = -J^{0i}$ are the Lorentz boosts. Therefore

$$J_i = \frac{1}{2}(L_i - iK_i) \tag{42}$$

Bear in mind that both $\frac{1}{2}(L_i - iK_i)$ and $\frac{1}{2}(L_i + iK_i)$ obey the $su(2)$ algebra.[13] Actually, the boost charges are vanishing as can be seen from Eq. (30). Thus we can obtain the angular momentum in the self-dual simple supergravity once J_{MN} is known.

We make a few final remarks. The total charges take the same integral form as those in the nonsupersymmetric case. Though we can obtain the $SU(2)$ sector of the $SL(2, \mathbb{C})$ charges, the information of the angular momentum is completely contained in the $SU(2)$ charges. It can be seen from the surface integrals that the angular momentum is governed by the r^{-2} part of $\tilde{e}^i$. As in refs. 1 and 2, we always assume that the phase space variables are subject to the boundary conditions

$$e^{\mu}_{AB}|_{\partial\Sigma} = \left(1 + \frac{M(\theta, \phi)}{r}\right)^2 \mathring{e}_{AB}{}^{\mu} + O(1/r^2), \qquad A_{\mu MN}|_{\partial\Sigma} = O(1/r^2) \tag{43}$$

$$\tilde{\pi}^i_A = O(1/r), \qquad \psi_{\mu A} = O(1/r) \tag{44}$$

where $\mathring{e}_{AB}{}^{\mu}$ denote the flat $SU(2)$ soldering forms. As a consequence, under the $SL(2, \mathbb{C})$ transforms behaving as

$$L^B_A = \Lambda^B_A + O(1/r^{1+\epsilon}) \qquad (\epsilon > 0) \tag{45}$$

where Λ are rigid transforms, the charges transform as

$$J_{MN} \to \Lambda^A_M \Lambda^B_N J_{AB} \tag{46}$$

i.e., they are gauge covariant. Their conservation is generally covariant. As in the nonsupersymmetric case,[7,8] the currents also have potentials, i.e., can be expressed as a divergence of an antisymmetric tensor density. So they are identically conserved. Upon quantization, the Poisson brackets correspond to the quantal commutators and their algebra realizes indeed the $su(2)$ algebra. This shows that the interpretations are reasonable.

It is novel that the relation between J_{MN} and the constraint $\mathcal{T}^{AB}$ is the same as that between the electric charge and the Gauss law constraint in QED,[14]

$$\nabla \cdot \mathbf{E} - e\bar{\psi}\gamma_0\psi = 0 \tag{47}$$

$$q = \int_{\partial\Sigma} \mathbf{E} \cdot d\mathbf{S} \tag{48}$$

So the J_{MN} is a kind of gauge charge.

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